

LIFTING ALGEBRAIC CONTRACTIONS IN C^* -ALGEBRAS

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ABSTRACT. Let p be a polynomial in one variable. It is shown that the universal C^* -algebra of the relation $p(x) = 0$, $\|x\| \leq C$ is semiprojective, residually finite-dimensional and has trivial extension group.

INTRODUCTION

Notions of projectivity and semiprojectivity for C^* -algebras was introduced by Effros and Kaminker [3] and, in its modern form, by Blackadar [1] as noncommutative analogues of absolute retract and absolute neighborhood retract in topology.

A C^* -algebra D is *projective* if for any C^* -algebra A , its ideal I and every $*$ -homomorphism $\phi : D \rightarrow A/I$, there exists a $*$ -homomorphism $\tilde{\phi}$ such that the diagram

$$\begin{array}{ccc} & & A \\ & \nearrow \tilde{\phi} & \downarrow \\ D & \xrightarrow{\phi} & A/I \end{array}$$

commutes.

A C^* -algebra D is *semiprojective* if for any C^* -algebra A , any increasing chain of ideals $I_1 \subseteq I_2 \subseteq \dots$ in A and for every $*$ -homomorphism $\phi : D \rightarrow A/\overline{\bigcup_k I_k}$, there exist n and a $*$ -homomorphism $\tilde{\phi} : D \rightarrow A/I_n$ such that the diagram

$$\begin{array}{ccc} & & A/I_n \\ & \nearrow \tilde{\phi} & \downarrow \\ D & \xrightarrow{\phi} & A/\overline{\bigcup_k I_k} \end{array}$$

commutes.

A notion of weak semiprojectivity was introduced by Eilers and Loring [4].

A C^* -algebra D is *weakly semiprojective* if for any sequence of C^* -algebras A_i and any $*$ -homomorphism $\phi : D \rightarrow \prod A_i / \bigoplus A_i$, there exists a $*$ -homomorphism $\tilde{\phi} : D \rightarrow \prod A_i$ such that the diagram

$$\begin{array}{ccc} & & \prod A_i \\ & \nearrow \tilde{\phi} & \downarrow \\ D & \xrightarrow{\phi} & \prod A_i / \bigoplus A_i \end{array}$$

commutes.

Examples and basic properties of projective and (weakly) semiprojective C^* -algebras can be found in [7].

Date: November 29, 2011.

2000 Mathematics Subject Classification. 46L05; 46L35.

Key words and phrases. Projective and semiprojective C^* -algebras, stable relation, lifting problem.

This work was partially supported by a grant from the Simons Foundation (208723 to Loring) and by the NordForsk Research Network "Operator Algebras and Dynamics" (grant 11580) .

The notion of (weak) (semi)projectivity provides an algebraic setting of lifting and perturbation problems for relations in C^* -algebras. Namely a relation is liftable (which means that in any quotient C^* -algebra A/I any elements satisfying the relation have preimages in A also satisfying the relation) if and only if the universal C^* -algebra of the relation is projective.

Similarly a relation is stable under small perturbations if and only if the universal C^* -algebra of the relation is weakly semiprojective.

The only problem is that not all relations have the universal C^* -algebras. However if relations are noncommutative $*$ -polynomial equations combined with norm restrictions on generators

$$\|x_i\| \leq c_i$$

then such system of relations defines the universal C^* -algebra. That is why in (weak) (semi)projectivity questions it is important to solve lifting problems for relations combined with norm restrictions on generators.

It is not easy for a polynomial relation to be liftable. For example, let us consider a polynomial in one variable. Suppose in a C^* -quotient we have an element x satisfying $p(x) = 0$. If p has a non-zero root, the spectral idempotent of x on this root also belongs to the quotient. But idempotents are not liftable: consider for example the unit of $\mathbb{C} = C_0(0, 1]/C_0(0, 1)$. It has no idempotent lift because $C_0(0, 1]$ contains no non-zero idempotents. Thus only monomials $x^n = 0$ have a chance to be liftable. And they indeed are liftable by deep result of Olsen and Pedersen [10]. Immediately there arises a question ([7]) if the universal C^* -algebra of $x^n = 0$, $\|x\| \leq C$ is projective. It was answered positively in [12] and in this paper we will give a short proof of that (Corollary 3).

As to non-monomial relations $p(x) = 0$, they are known to be stable under small perturbations [5]. Moreover the proof in [5] can be generalized to show that relations $p(x) = 0$ are liftable from quotients of the form $A/\overline{\bigcup I_n}$ arising in definition of semiprojectivity, and one is led to ask if the universal C^* -algebra of $p(x) = 0$, $\|x\| \leq C$ is semiprojective. In [13] it was proved in the case when all roots of the polynomial have multiplicity more than 1. In this paper we prove it for arbitrary polynomial (Theorem 9).

We also show that these universal C^* -algebras are RFD (Theorem 10) and that $*$ -homomorphisms from these C^* -algebras to Calkin algebra lift to $*$ -homomorphisms to $B(H)$ (Theorem 11). The last result is generalization of Olsen's structure theorem for polynomially compact operators [11].

Our main technical tool is a generalized spectral radius formula we introduced in [9] in connection with question of Olsen about best approximation of operators by compacts. It turns out to be useful tool also for lifting polynomial relations combined with restrictions on norms of generators.

A GENERALIZED SPECTRAL RADIUS FORMULA

For $x \in A$, we denote by \dot{x} its image in A/I and by $\rho(x)$ its spectral radius.

The following theorem is a generalization of spectral radius formula of Murphy and West [8]. The spectral radius formula is particular case of the generalized spectral radius formula when $I = A$.

Theorem 1. ([9]) *Let A be a C^* -algebra, I its ideal, $x \in A$. Then*

$$\max\{\rho(x), \|\dot{x}\|\} = \inf \|(1+i)x(1+i)^{-1}\|$$

(here \inf is taken over all $i \in I$ such that $1+i$ is invertible). If $\|\dot{x}\| > \rho(x)$ then the infimum in the right-hand side is attained.

Lemma 2. *Let p be a polynomial in one variable and t_1, \dots, t_k its roots. Let A be a C^* -algebra, I its ideal, $x \in A/I$, $p(x) = 0$ and $\|x\| > \max\{t_i\}$. Suppose x has a lift $X \in A$ such that $p(X) = 0$. Then there is a lift $\tilde{X} \in A$ of x such that $p(\tilde{X}) = 0$ and $\|\tilde{X}\| = \|x\|$.*

Proof. Since $p(X) = 0$,

$$\rho(X) = \max\{t_i\} < \|x\|.$$

By the generalized spectral radius formula there exists $i \in I$ such that

$$\|(1+i)X(1+i)^{-1}\| = \|x\|.$$

Let $\tilde{X} = (1+i)X(1+i)^{-1}$. Then \tilde{X} is a lift of x , $\|\tilde{X}\| = \|x\|$ and

$$p(\tilde{X}) = p((1+i)X(1+i)^{-1}) = (1+i)p(X)(1+i)^{-1} = 0.$$

□

Corollary 3. ([12]) *The universal C^* -algebra*

$$C^*\langle x \mid x^n = 0, \|x\| \leq C \rangle$$

is projective.

Proof. For $C = 0$ the statement obviously holds. So let $C > 0$ and let $x \in A/I$, $x^n = 0$, $\|x\| \leq C$. We need to show that there a lift of x with the same properties. If $x = 0$ then it is obvious, so let us assume $x \neq 0$. By [10] there is a lift X of x such that $X^n = 0$. By Lemma 2 there is a lift \tilde{X} of x such that $(\tilde{X})^n = 0$ and $\|\tilde{X}\| = \|x\| \leq C$. □

SEMIPROJECTIVITY OF THE UNIVERSAL C^* -ALGEBRA $C^*\langle x \mid p(x) = 0, \|x\| \leq C \rangle$

Lemma 4. *Let $T \in B(H)$ and $(T - t_N)^{k_N}(T - t_{N-1})^{k_{N-1}} \dots (T - t_1)^{k_1} = 0$. Let*

$$\begin{aligned} H_1 &= \ker(T - t_1) \\ H_2 &= \ker(T - t_1)^2 \ominus H_1 \\ &\dots \\ H_{k_1} &= \ker(T - t_1)^{k_1} \ominus H_{k_1-1} \\ H_{k_1+1} &= \ker(T - t_2)(T - t_1)^{k_1} \ominus H_{k_1} \\ &\dots \\ H_{k_1+\dots+k_N} &= \ker(T - t_N)^{k_N-1}(T - t_{N-1})^{k_{N-1}-1} \dots (T - t_1)^{k_1} \ominus H_{k_1+\dots+k_N-1}. \end{aligned}$$

Then with respect to the decomposition $H = H_1 \oplus \dots \oplus H_{k_1+\dots+k_N}$ the operator T has uppertriangular form with $t_1 1, \dots, t_1 1, \dots, t_N 1, \dots, t_N 1$ on the diagonal, where each $t_i 1$ is repeated k_i times.

Proof. If $x \in H_1$, then $Tx = t_1 x$. If $x \in H_2$, then $Tx = (T - t_1)x + t_1 x$, where $(T - t_1)x \in H_1$. And so on. □

Lemma 5. *Let $B \subseteq B(H)$ be a C^* -algebra, $b \in B$ an idempotent. Then the projection onto the range of b also belongs to B .*

Proof. By Lemma 4, b can be written as

$$b = \begin{pmatrix} 1 & X \\ 0 & 0 \end{pmatrix}. \quad (1)$$

Hence

$$\begin{pmatrix} 1 + XX^* & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & X \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ X^* & 0 \end{pmatrix} = bb^* \in B.$$

Let f be a continuous function on \mathbb{R} which vanishes at 0 and equal 1 at $[1, \infty)$. Then

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = f \left(\begin{pmatrix} 1 + XX^* & 0 \\ 0 & 0 \end{pmatrix} \right) \in B.$$

From (1) it is seen that $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is exactly the projection onto the range of b . \square

Let L_1 be direct sum of first k_1 summands in the decomposition $H = H_1 \oplus \dots \oplus H_{k_1+\dots+k_N-1}$ (that is $\ker(T - t_1)^{k_1}$), L_2 be direct sum of next k_2 summands, and so on. For any m , let $M_m = H \ominus (L_1 \oplus \dots \oplus L_m)$.

Corollary 6. *Let $T \in B(H)$,*

$$(T - t_N)^{k_N} (T - t_{N-1})^{k_{N-1}} \dots (T - t_1)^{k_1} = 0$$

and subspaces L_i be as above. Then projections onto L_i belong to $C^(T, 1)$.*

Proof. By transposing factors in the product $(T - t_N)^{k_N} (T - t_{N-1})^{k_{N-1}} \dots (T - t_1)^{k_1}$ the general case can be reduced to the case $i = 1$. So let us prove that the projection onto L_1 belongs to $C^*(T, 1)$. Since t_1 is an isolated point of $\sigma(T)$, there exists the spectral idempotent Q corresponding to t_1 , that is

$$Q = \chi(T),$$

where χ is equal to 1 in a neighborhood of t_1 and is equal to zero in a neighborhood of $\sigma(T) \setminus \{t_1\}$. By Lemma 5, it is sufficient to prove that $\text{Ran } Q = L_1$.

With respect to the decomposition $H = L_1 \oplus M_1$ the operator T is of the form

$$T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix},$$

where $\sigma(A) = \{t_1\}$, $\sigma(C) = \sigma(T) \setminus \{t_1\}$. Hence

$$Q = \chi(T) = \begin{pmatrix} \chi(A) & * \\ 0 & \chi(C) \end{pmatrix} = \begin{pmatrix} 1 & * \\ 0 & 0 \end{pmatrix}.$$

Hence $\text{Ran } Q = L_1$. \square

Lemma 7. *Let $A \in B(H)$ be given by $A = (A_{ij})$ with respect to some orthonormal basis $\{e_i\}$ in H . If $|A_{11}| = \|A\|$, then $A_{1j} = A_{j1} = 0$, when $j \neq 1$.*

Proof. We have

$$(AA^*)_{11} = \sum_{j \geq 1} |A_{1j}|^2 = \|A\|^2 + \sum_{j > 1} |A_{1j}|^2.$$

Since

$$(AA^*)_{11} = (AA^*e_1, e_1) \leq \|AA^*\| = \|A\|^2,$$

we get $A_{1j} = 0$, when $j > 1$. Applying this to A^* , we get $A_{j1} = 0$, when $j > 1$. \square

In what follows we will assume $|t_1| \geq |t_2| \geq \dots$ which always can be done by transposition of factors in the product.

Corollary 8. *Suppose $T \in B(H)$ and*

$$(T - t_N)^{k_N} (T - t_{N-1})^{k_{N-1}} \dots (T - t_1)^{k_1} = 0.$$

Then there exists $0 \leq m \leq N$ such that with respect to the decomposition $H = L_1 \oplus \dots \oplus L_m \oplus M_m$

$$T = t_1 1 \oplus \dots \oplus t_m 1 \oplus S,$$

where S is such that $(S - t_{m+1})^{k_{m+1}} \dots (S - t_N)^{k_N} = 0$ and $\|S\| > |t_i|$, for all $i \geq m + 1$.

Proof. We write T in upper-triangular form as in Lemma 4 and then use Lemma 7. \square

Let p be a polynomial in one variable, $C \geq 0$. Below the universal C*-algebra

$$\mathcal{A} = C^* \langle p(x) = 0, \|x\| \leq C \rangle$$

is denoted by \mathcal{A} .

Theorem 9. *\mathcal{A} is semiprojective.*

Proof. Write p as $p(x) = (x - t_N)^{k_N} \dots (x - t_1)^{k_1} = 0$. Let $b \in A/I$, $I = \overline{UI_n}$, $(b - t_N)^{k_N} \dots (b - t_1)^{k_1} = 0$, $\|b\| \leq C$. Embed A/I into $B(H)$ and write b as in Corollary 8. Let p_1, \dots, p_m be the projections onto L_1, \dots, L_m and p_{m+1} be the projection onto M_m . By Corollary 6, they all belong to A/I . Then

$$b = \sum_{i=1}^m t_i p_i + s,$$

where $s \in p_{m+1} A/I p_{m+1}$ satisfies the equation

$$(s - t_{m+1})^{k_{m+1}} \dots (s - t_N)^{k_N} = 0 \tag{2}$$

and

$$\|s\| > \max_{i \geq m+1} |t_i|. \tag{3}$$

By [1] there exists n such that p_i 's can be lifted to projections P_i 's in A/I_n with $\sum_{i=1}^{m+1} P_i = 1$. Since $A/I = (A/I_n)/(I/I_n)$, we have

$$p_{m+1} A/I p_{m+1} = (P_{m+1} A/I_n P_{m+1}) / (P_{m+1} I/I_n P_{m+1})$$

and, by (2), (3) and Lemma 2 we can lift s to $S \in P_{m+1} A/I_n P_{m+1}$ with

$$(S - t_{m+1})^{k_{m+1}} \dots (S - t_N)^{k_N} = 0, \quad \|S\| = \|s\| \leq C.$$

Let

$$a = \sum_{i=1}^m t_i P_i + S. \tag{4}$$

It is a lift of b , $\|a\| \leq C$ and $(a - t_N)^{k_N} \dots (a - t_1)^{k_1} = 0$. The last equality can be checked by direct calculations, but it is easier to say that (4) corresponds to upper-triangular form of a as in Corollary 8 and then the last equality follows instantly. \square

Theorem 10. *\mathcal{A} is RFD.*

Proof. Let $H = l^2(\mathbb{N})$. We will identify the algebra M_n of n -by- n matrices with

$$B(l^2\{1, \dots, n\}) \subseteq B(H).$$

Let $\mathcal{B} \subseteq \prod M_n$ be the C^* -algebra of all $*$ -strongly convergent sequences and let \mathcal{I} be the ideal of all sequences $*$ -strongly convergent to zero. Then we can identify \mathcal{B}/\mathcal{I} with $B(H)$ by sending each sequence to its $*$ -strong limit.

We claim that any family p_1, \dots, p_n of projections with sum 1 in $B(H)$ lifts to a family of projections P_1, \dots, P_n with sum 1 in \mathcal{B} . One way to prove this is to modify the argument of Choi, used the proof of Theorem 7 of [2]. A more modern approach is to use Hadwin's [6] characterization of separable RFD C^* -algebras: in the unital separable case, D is RFD if and only if we can lift all elements of $\text{hom}_1(D, B(H))$ to $\text{hom}_1(D, \mathcal{B})$. Clearly \mathbb{C}^n is RFD and this lifting problem for \mathbb{C}^n is equivalent to the needed lift of n projections that sum to the identity.

Let $\pi : \mathcal{A} \rightarrow B(H)$ be the universal representation. Arguments from the proof of Theorem 9 can be repeated without any change to show that π lifts to a $*$ -homomorphism $\tilde{\pi} : \mathcal{A} \rightarrow \mathcal{B}$. This lift gives a separating family of finite-dimensional representations. \square

Theorem 11. *Any $*$ -homomorphism from \mathcal{A} to Calkin algebra lifts to a $*$ -homomorphism to $B(H)$. In particular $\text{Ext}(\mathcal{A}) = 0$.*

Proof. As is well known, orthogonal projections (with sum 1) in Calkin algebra can be lifted to orthogonal projections (with sum 1) in $B(H)$. Now we can repeat arguments from the proof of Theorem 9. \square

Remark 12. *To each x in any C^* -algebra with $p(x) = 0$ we can assign in a canonical and functorial way a collection of projections that are orthogonal and sum to one. See Corollary 6, or [5]. If $x \in A/I$ and these projections lift, then x lifts, preserving the relation $p(x) = 0$ and the norm. In formal terms,*

$$\mathbb{C}^{N-1} \rightarrow C^* \langle x \mid p(x) = 0, \|x\| \leq C \rangle$$

is conditionally projective. Thus we have improved upon Theorem 2 in [5] by incorporating the norm condition.

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